

RENORMALIZATION METHOD IN p -ADIC λ -MODEL ON THE CAYLEY TREE

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ABSTRACT. In this present paper, it is proposed the renormalization techniques in the investigation of phase transition phenomena in p -adic statistical mechanics. We mainly study p -adic λ -model on the Cayley tree of order two. We consider generalized p -adic quasi Gibbs measures depending on parameter $\rho \in \mathbb{Q}_p$, for the λ -model. Such measures are constructed by means of certain recurrence equations. These equations define a dynamical system. We study two regimes with respect to parameters. In the first regime we establish that the dynamical system has one attractive and two repelling fixed points, which predicts the existence of a phase transition. In the second regime the system has two attractive and one neutral fixed points, which predicts the existence of a quasi phase transition. A main point of this paper is to verify (i.e. rigorously prove) and confirm that the indicated predictions (via dynamical systems point of view) are indeed true. To establish the main result, we employ the methods of p -adic analysis, and therefore, our results are not valid in the real setting.

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Key words: p -adic numbers, λ -model; p -adic quasi Gibbs measure, strong phase transition, Cayley tree.

1. INTRODUCTION

After Wilsons seminal work in the early 1970's [61], based also on the ground breaking foundations laid by Kadanoff, Widom, Michael Fisher [14], and others in the preceding decade, the renormalization group (RG) has had a profound impact on modern statistical physics. Not only do renormalization group techniques provide a powerful tool to analytically describe and quantitatively capture both static and dynamic critical phenomena near continuous phase transitions that are governed by strong interactions, fluctuations, and correlations. RG presents a conceptual framework and mathematical language that has become ubiquitous in the theoretical description of many complex interacting many-particle systems encountered in nature (see for review [57]).

The renormalization method is then applied in statistical mechanics and yielded lots of interesting results. Since such investigations of phase transitions of spin models on hierarchical lattices showed that they make the exact calculation of various physical quantities [5, 18]. One of the simplest hierarchical lattice is Cayley tree or Bethe lattice (see [50]). This lattice is not a realistic lattice, however, investigations of phase transitions of spin models on trees like Cayley tree showed that they make the exact calculation of various physical quantities [53]. It is believed that several among its interesting thermal properties could persist for regular lattices, for which the exact calculation is far intractable. Illustrations of the renormalization methods are widely shown in the study of Ising model [5], since it has wide theoretical interest and practical applications. Therefore, one of generalizations of the Ising model is so-called λ -model on the Cayley tree (see [52, 38]). Such a model has enough rich structure to illustrate almost

every conceivable nuance of statistical mechanics. We have to stress that one of the central problems in the theory of Gibbs measures of lattice systems is to describe infinite-volume (or limiting) Gibbs measures corresponding to a given Hamiltonian. A complete analysis of this set is often a difficult problem (see for review [18, 53]).

On the other hand, there are many investigations have been done to discuss and debate the question due to the assumption that p -adic numbers provide a more exact and more adequate description of microworld phenomena (see for example [10, 56, 59]). Therefore, starting the 1980s, various models described in the language of p -adic analysis have been actively studied [3],[15],[36]. The well-known studies in this area are primarily devoted to investigating quantum mechanics models using equations of mathematical physics [4, 2, 28, 29, 60, 58]. We refer the reader to [12] for recent development of the subject.

One of the first applications of p -adic numbers in quantum physics appeared in the framework of quantum logic in [6]. This model is especially interesting for us because it could not be described by using conventional real valued probability (see [29, 34, 36, 58]). Therefore, p -adic probability models were investigated in [27, 32, 33]. Using that p -adic measure theory in [30, 31, 35], the theory of p -adic and non-Archimedean stochastic processes has been developed. These investigations allowed us to construct wide classes of stochastic processes using finite dimensional probability distributions [17]. In [16],[39]-[45],[48, 49] it has been developed p -adic statistical mechanics within the scheme of the theory of p -adic probability and p -adic stochastic processes. Namely, we have studied p -adic Ising and Potts models with nearest neighbor interactions on Cayley trees.

In the present paper, we propose to study phase transition phenomena of p -adic statistical models by means of renormalization methods in the measure-theoretical scheme. Note that the renormalization method is closely related to the investigation of dynamical system associated with a given model. Therefore, in what follows, methods of p -adic dynamical systems and p -adic probability measures will be used. In this paper, we illustrate our propose in the study of p -adic λ -model which was started in [25, 26]. In this model spin takes two different values. In the mentioned papers we studied only the uniqueness of p -adic Gibbs measures of the model. Recently, in [40, 41] it was introduced two kind of notions of phase transition: *phase transition* and *quasi phase transition*. Note that the investigate of phase transitions by dynamical system approach, in real case, has greatly enhanced understanding of complex properties of models. The interplay of statistical mechanics with chaos theory has even led to novel conceptual frameworks in different physical settings [13]. Therefore, a main aim of this paper is to apply and verify renormalization method to the existence of phase transitions.

Let us highlight the organization of the paper. In section 2 we collect necessary definitions and preliminary results which will be used in the paper. In section 3 we provide a measure-theoretical construction of generalized p -adic quasi Gibbs measures for the λ -model. Such kind of measures exist if the interacting functions satisfy certain recurrence equation. In section 4 we consider two regimes with respect to a parameters A and C . In this section we prove the existence of generalized p -adic Gibbs measures in both regimes. The obtained recurrence equations define a dynamical system. In the first regime we establish that the dynamical system has one attractive and two repelling fixed points, which predicts the existence of a phase transition. In the second regime the system has two attractive and one neutral fixed points, which predicts the existence of a quasi phase transition. In section 5, we verify (i.e.

rigorously prove) and confirm that the indicated predictions (via dynamical systems point of view) are indeed true. To establish the main result, we employ the methods of p -adic analysis, and therefore, our results are not valid in the real setting.

2. PRELIMINARIES

2.1. p -adic numbers. In what follows p will be a fixed prime number. The set \mathbb{Q}_p is defined as a completion of the rational numbers \mathbb{Q} with respect to the norm $|\cdot|_p : \mathbb{Q} \rightarrow \mathbb{R}$ given by

$$(2.1) \quad |x|_p = \begin{cases} p^{-r} & x \neq 0, \\ 0, & x = 0, \end{cases}$$

here, $x = p^r \frac{m}{n}$ with $r, m \in \mathbb{Z}$, $n \in \mathbb{N}$, $(m, p) = (n, p) = 1$. The absolute value $|\cdot|_p$ is non-Archimedean, meaning that it satisfies the *strong triangle inequality* $|x + y|_p \leq \max\{|x|_p, |y|_p\}$. We recall a nice property of the norm, i.e. if $|x|_p > |y|_p$ then $|x + y|_p = |x|_p$. Note that this is a crucial property which is proper to the non-Archimedeanity of the norm.

Any p -adic number $x \in \mathbb{Q}_p$, $x \neq 0$ can be uniquely represented in the form

$$(2.2) \quad x = p^{\gamma(x)}(x_0 + x_1p + x_2p^2 + \dots),$$

where $\gamma = \gamma(x) \in \mathbb{Z}$ and x_j are integers, $0 \leq x_j \leq p - 1$, $x_0 > 0$, $j = 0, 1, 2, \dots$. In this case $|x|_p = p^{-\gamma(x)}$.

We recall that an integer $a \in \mathbb{Z}$ is called a *quadratic residue modulo p* if the equation $x^2 \equiv a \pmod{p}$ has a solution $x \in \mathbb{Z}$.

Lemma 2.1. [34] *In order that the equation*

$$x^2 = a, \quad 0 \neq a = p^{\gamma(a)}(a_0 + a_1p + \dots), \quad 0 \leq a_j \leq p - 1, \quad a_0 > 0$$

has a solution $x \in \mathbb{Q}_p$, it is necessary and sufficient that the following conditions are fulfilled:

- (i) $\gamma(a)$ is even;
- (ii) a_0 is a quadratic residue modulo p if $p \neq 2$, and moreover $a_1 = a_2 = 0$ if $p = 2$.

For each $a \in \mathbb{Q}_p$, $r > 0$ we denote

$$B(a, r) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}, \quad \mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}.$$

Recall that the p -adic exponential is defined by

$$\exp_p(x) = \sum_{n=1}^{\infty} \frac{x^n}{n!},$$

which converges for every $x \in B(0, p^{-1/(p-1)})$. It is known [34] that for any $x \in B(0, p^{-1/(p-1)})$ one has

$$|\exp_p(x)|_p = 1, \quad |\exp_p(x) - 1|_p = |x|_p < 1.$$

Put

$$(2.3) \quad \mathcal{E}_p = \{x \in \mathbb{Q}_p : |x|_p = 1, \quad |x - 1|_p < p^{-1/(p-1)}\}.$$

Note that the basics of p -adic analysis, p -adic mathematical physics are explained in [34, 54, 58].

Now we recall some standard terminology of the theory of dynamical systems (see for example [1],[32]).

Let (f, B) be a dynamical system in \mathbb{Q}_p , where $f : x \in B \rightarrow f(x) \in B$ is some function and $B = B(a, r)$ or \mathbb{Q}_p . Denote $x^{(n)} = f^n(x^{(0)})$, where $x^0 \in B$ and $f^n(x) = \underbrace{f \circ \cdots \circ f}_n(x)$. If $f(x^{(0)}) = x^{(0)}$ then $x^{(0)}$ is called a *fixed point*. Let $x^{(0)}$ be a fixed point of an analytic function $f(x)$. Set

$$\lambda = \frac{d}{dx}f(x^{(0)}).$$

The point $x^{(0)}$ is called *attractive* if $0 \leq |\lambda|_p < 1$, *neutral* if $|\lambda|_p = 1$, and *repelling* if $|\lambda|_p > 1$.

It is known [32] that if a fixed point $x^{(0)}$ is attractive then there exists a neighborhood $U(x^{(0)}) (\subset B)$ of $x^{(0)}$ such that for all points $y \in U(x^{(0)})$ it holds $\lim_{n \rightarrow \infty} f^n(y) = x^{(0)}$. If a fixed point $x^{(0)}$ is repelling, then there exists a neighborhood $U(x^{(0)})$ of $x^{(0)}$ such that $|f(x) - x^{(0)}|_p > |x - x^{(0)}|_p$ for $x \in U(x^{(0)})$, $x \neq x^{(0)}$.

2.2. p -adic measure. Let (X, \mathcal{B}) be a measurable space, where \mathcal{B} is an algebra of subsets X . A function $\mu : \mathcal{B} \rightarrow \mathbb{Q}_p$ is said to be a *p -adic measure* if for any $A_1, \dots, A_n \subset \mathcal{B}$ such that $A_i \cap A_j = \emptyset$ ($i \neq j$) the equality holds

$$\mu\left(\bigcup_{j=1}^n A_j\right) = \sum_{j=1}^n \mu(A_j).$$

A p -adic measure is called a *probability measure* if $\mu(X) = 1$. One of the important condition (which was already invented in the first Monna–Springer theory of non-Archimedean integration [37]) is boundedness, namely a p -adic probability measure μ is called *bounded* if $\sup\{|\mu(A)|_p : A \in \mathcal{B}\} < \infty$. We pay attention to an important special case in which boundedness condition by itself provides a fruitful integration theory (see for example [30]). Note that, in general, a p -adic probability measure need not be bounded [51]. For more detail information about p -adic measures we refer to [27, 32, 51].

2.3. Cayley tree. Let $\Gamma_+^k = (V, L)$ be a semi-infinite Cayley tree of order $k \geq 1$ with the root x^0 (whose each vertex has exactly $k + 1$ edges, except for the root x^0 , which has k edges). Here V is the set of vertices and L is the set of edges. The vertices x and y are called *nearest neighbors* and they are denoted by $l = \langle x, y \rangle$ if there exists an edge connecting them. A collection of the pairs $\langle x, x_1 \rangle, \dots, \langle x_{d-1}, y \rangle$ is called a *path* from the point x to the point y . The distance $d(x, y)$, $x, y \in V$, on the Cayley tree, is the length of the shortest path from x to y .

$$W_n = \{x \in V \mid d(x, x^0) = n\}, \quad V_n = \bigcup_{m=0}^n W_m, \quad L_n = \{l = \langle x, y \rangle \in L \mid x, y \in V_n\}.$$

The set of direct successors of x is defined by

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, x \in W_n.$$

Observe that any vertex $x \neq x^0$ has k direct successors and x^0 has $k + 1$.

Given a set A , by $|A|$ we denote the number of its elements. In what follows, we need the following equalities:

$$(2.4) \quad |W_n| = k^n, \quad |V_n| = \frac{k^{n+1} - 1}{k - 1},$$

$$(2.5) \quad |W_n| = (k - 1)|V_{n-1}| + 1, \quad |V_n| = k|V_{n-1}| + 1.$$

3. p -ADIC λ MODEL AND ITS p -ADIC QUASI GIBBS MEASURES

In this section we consider the p -adic λ -model where spin takes values in the set $\Phi = \{-1, +1\}$, (Φ is called a *state space*) and is assigned to the vertices of the tree $\Gamma_+^k = (V, L)$. A configuration σ on V is then defined as a function $x \in V \rightarrow \sigma(x) \in \Phi$; in a similar manner one defines configurations σ_n and ω on V_n and W_n , respectively. The set of all configurations on V (resp. V_n , W_n) coincides with $\Omega = \Phi^V$ (resp. $\Omega_{V_n} = \Phi^{V_n}$, $\Omega_{W_n} = \Phi^{W_n}$). One can see that $\Omega_{V_n} = \Omega_{V_{n-1}} \times \Omega_{W_n}$. Using this, for given configurations $\sigma_{n-1} \in \Omega_{V_{n-1}}$ and $\omega \in \Omega_{W_n}$ we define their concatenations by

$$(\sigma_{n-1} \vee \omega)(x) = \begin{cases} \sigma_{n-1}(x), & \text{if } x \in V_{n-1}, \\ \omega(x), & \text{if } x \in W_n. \end{cases}$$

It is clear that $\sigma_{n-1} \vee \omega \in \Omega_{V_n}$.

Assume for each edge $\langle x, y \rangle \in L$ a function $\lambda : \Phi \times \Phi \rightarrow \mathbb{Z}$ is given. Then the Hamiltonian $H_n : \Omega_{V_n} \rightarrow \mathbb{Z}$ of the p -adic λ -model is defined by

$$(3.1) \quad H_n(\sigma) = \sum_{\langle x, y \rangle \in L_n} \lambda(\sigma(x), \sigma(y))$$

Remark 3.1. This model first has been considered in [25]. In the real setting such kind of model was studied in [52]. We remark that if one takes $\lambda(u, v) = Nuv$ for some integer N , then the model (3.1) reduces to the well-known Ising model (see [17, 25, 39]).

Let $\rho \in \mathbb{Q}_p$ and assume that $\mathbf{h} : x \in V \setminus \{x^{(0)}\} \rightarrow h_x \in \mathbb{Q}_p$ be a mapping. Given $n \in \mathbb{N}$, let us consider a p -adic probability measure $\mu_{\mathbf{h}}^{(n)}$ on Ω_{V_n} defined by

$$(3.2) \quad \mu_{\mathbf{h}, \rho}^{(n)}(\sigma) = \frac{1}{Z_{n, \rho}^{(\mathbf{h})}} \rho^{H_n(\sigma)} \prod_{x \in W_n} (h_x)^{\sigma(x)}$$

Here, $\sigma \in \Omega_{V_n}$, and $Z_{n, \rho}^{(\mathbf{h})}$ is the corresponding normalizing factor called a *partition function* given by

$$(3.3) \quad Z_{n, \rho}^{(\mathbf{h})} = \sum_{\sigma \in \Omega_{V_n}} \rho^{H_n(\sigma)} \prod_{x \in W_n} (h_x)^{\sigma(x)}.$$

We recall [40] that one of the central results of the theory of probability concerns a construction of an infinite volume distribution with given finite-dimensional distributions, which is a well-known *Kolmogorov's extension Theorem* [55]. Recall that a p -adic probability measure μ on Ω is *compatible* with defined ones $\mu_{\mathbf{h}}^{(n)}$ if one has

$$(3.4) \quad \mu(\sigma \in \Omega : \sigma|_{V_n} = \sigma_n) = \mu_{\mathbf{h}, \rho}^{(n)}(\sigma_n), \quad \text{for all } \sigma_n \in \Omega_{V_n}, \quad n \in \mathbb{N}.$$

The existence of the measure μ is guaranteed by the p -adic Kolmogorov's Theorem [17, 31]. Namely, if the measures $\mu_{\mathbf{h},\rho}^{(n)}$, $n \geq 1$ satisfy the *compatibility condition*, i.e.

$$(3.5) \quad \sum_{\omega \in \Omega_{W_n}} \mu_{\mathbf{h},\rho}^{(n)}(\sigma_{n-1} \vee \omega) = \mu_{\mathbf{h},\rho}^{(n-1)}(\sigma_{n-1}),$$

for any $\sigma_{n-1} \in \Omega_{V_{n-1}}$, then there is a unique measure μ on Ω with (3.4).

Now following [40] if for some function \mathbf{h} the measures $\mu_{\mathbf{h},\rho}^{(n)}$ satisfy the compatibility condition, then there is a unique p -adic probability measure, which we denote by $\mu_{\mathbf{h},\rho}$, since it depends on \mathbf{h} and ρ . Such a measure $\mu_{\mathbf{h}}$ is said to be a *generalized p -adic quasi Gibbs measure* corresponding to the p -adic λ -model. By $Q\mathcal{G}(H)$ we denote the set of all generalized p -adic quasi Gibbs measures associated with functions $\mathbf{h} = \{\mathbf{h}_x, x \in V\}$. If there are at least two distinct generalized p -adic quasi Gibbs measures $\mu, \nu \in Q\mathcal{G}(H)$ such that μ is bounded and ν is unbounded, then we say that a *phase transition* occurs. By another words, one can find two different functions \mathbf{s} and \mathbf{h} defined on \mathbb{N} such that there exist the corresponding measures $\mu_{\mathbf{s},\rho}$ and $\mu_{\mathbf{h},\rho}$, for which one is bounded, another one is unbounded. Moreover, if there is a sequence of sets $\{A_n\}$ such that $A_n \in \Omega_{V_n}$ with $|\mu(A_n)|_p \rightarrow 0$ and $|\nu(A_n)|_p \rightarrow \infty$ as $n \rightarrow \infty$, then we say that there occurs a *strong phase transition*. If there are two different functions \mathbf{s} and \mathbf{h} defined on \mathbb{N} such that there exist the corresponding measures $\mu_{\mathbf{s},\rho}$, $\mu_{\mathbf{h},\rho}$, and they are bounded, then we say there is a *quasi phase transition*.

Note that some comparison of these phase transitions with real counterparts was highlighted in [40]. In [40, 43] the existence of the strong phase transition for the $q + 1$ -state Potts model on the Cayley tree has been proved. In the present paper, we are going to establish such kind of phenomena for the λ -model.

One can prove the following theorem.

Theorem 3.1. *The measures $\mu_{\mathbf{h},\rho}^{(n)}$, $n = 1, 2, \dots$ (see (3.2)), associated with λ -model (3.1), satisfy the compatibility condition (3.5) if and only if for any $x \in V \setminus \{x^{(0)}\}$ the following equation holds:*

$$(3.6) \quad h_x^2 = \prod_{y \in S(x)} \left(\frac{\rho^{\lambda(1,1)} h_y^2 + \rho^{\lambda(1,-1)}}{\rho^{\lambda(-1,1)} h_y^2 + \rho^{\lambda(-1,-1)}} \right).$$

The proof can be proceeded by the same argument as in [25].

According to Theorem 3.1 the problem of describing the generalized p -adic quasi Gibbs measures is reduced to the description of solutions of the functional equations (3.6).

4. DYNAMICAL SYSTEM AND THE EXISTENCE OF GENERALIZED p -ADIC QUASI GIBBS MEASURES

In this section we consider the λ -model (3.1) over the Cayley tree of order two, i.e. $k = 2$. Main aim of this section is to establish the existence of generalized p -adic quasi Gibbs measures by analyzing the equation (3.6). In the sequel, we will consider a case when $|\rho|_p < 1$ and $p \geq 3$. Note that the case $\rho \in \mathcal{E}_p$ has been studied in [25, 44].

Recall that a function $\mathbf{h} = \{\mathbf{h}_x\}_{x \in V \setminus \{x^0\}}$ is called *translation-invariant* if $\mathbf{h}_x = \mathbf{h}_y$ for all $x, y \in V$. A p -adic measure $\mu_{\mathbf{h}}$, corresponding to a translation-invariant function \mathbf{h} , is called a *translation-invariant generalized p -adic quasi Gibbs measure*.

To solve the equation (3.6), in general, is very complicated. Therefore, let us first restrict ourselves to the description of translation-invariant solutions of (3.6). More exactly, we suppose that $h_x := h$ for all $x \in V$. Then from (3.6) we find

$$(4.1) \quad h^2 = \left(\frac{Ah^2 + B}{Ch^2 + D} \right)^2,$$

where $A = \rho^{\lambda(1,1)}, B = \rho^{\lambda(1,-1)}, C = \rho^{\lambda(-1,1)}, D = \rho^{\lambda(-1,-1)}$.

The last equation splits into the following ones:

$$(4.2) \quad h = \left(\frac{Ah^2 + B}{Ch^2 + D} \right),$$

$$(4.3) \quad h = \left(\frac{Ah^2 + B}{Ch^2 + D} \right).$$

One can see that (4.3) is conjugate to (4.2) via $h(x) = -x$. Therefore, we will investigate the equation (4.2).

In this paper, we restrict ourselves to a special case. Namely, we assume that $|A|_p, |C|_p < 1$, $B = D = 1$, i.e. $\lambda(1,1), \lambda(-1,1) \in \mathbb{N}$, $\lambda(1,-1) = \lambda(-1,-1) = 0$. In what follows, we will assume that $|A|_p \neq |C|_p$, otherwise one finds $A = C$ and correspondingly equation (4.2) becomes trivial.

Let us denote

$$S = \{x \in \mathbb{Q}_p : |x|_p = 1\}.$$

Lemma 4.1. *Let $p \geq 3$, and $|A|_p, |C|_p < 1$ and f be given by*

$$(4.4) \quad f(x) = \frac{Ax^2 + 1}{Cx^2 + 1}.$$

Then $f(S) \subset S$ and

$$|f(x) - f(y)|_p \leq |A - C||x - y|_p,$$

for all $x, y \in S$.

Proof. Assume that $u \in S$. Then from

$$(4.5) \quad |Au^2 + 1|_p = |Cu^2 + 1|_p = 1$$

one gets $f(S) \subset S$. Now let us show the second condition. Let $x, y \in S$, then from (4.5) we have

$$\begin{aligned} |f(x) - f(y)|_p &= \left| \frac{C(y^2 - x^2) + A(x^2 - y^2)}{(Cx^2 + 1)(Cy^2 + 1)} \right|_p \\ &\leq |A - C|_p |x - y|_p. \end{aligned}$$

This completes the proof. □

Now we can formulate the following proposition about fixed points of f .

Theorem 4.2. *Let $|A|_p, |C|_p < 1$ with $|A|_p \neq |C|_p$, and f be given by (4.4). Then the following statements hold:*

- (i) *The function f has a unique fixed point x_0 in \mathcal{E}_p ;*

- (ii) Assume that $|A|_p^2 < |C|_p$. Then the function f has at most two fixed points x_1, x_2 different from x_0 if and only if $\sqrt{-C}$ exists; Moreover, one has

$$(4.6) \quad |x_{1,2}|_p = \frac{1}{\sqrt{|C|_p}};$$

- (iii) Assume that $|A|_p^2 > |C|_p$. Then the function f has two fixed points x_1, x_2 different from x_0 . Moreover, one has

$$(4.7) \quad |x_1|_p = \frac{|A|_p}{|C|_p}, \quad |x_2|_p = \frac{1}{|A|_p}.$$

Proof. (i) By Lemma 4.1 we conclude that f satisfies the Banach contraction principle on S . Therefore, there exists $x_0 \in S$ such that $f(x_0) = x_0$. Let us show that $x_0 \in \mathcal{E}_p$. Indeed, we have

$$\begin{aligned} |x_0 - 1|_p &= |f(x_0) - 1|_p = \left| \frac{Ax^2 + 1}{Cx^2 + 1} - 1 \right|_p \\ &= |(A - C)x_0^2|_p \\ (4.8) \quad &= |A - C|_p < 1 \end{aligned}$$

this means $x_0 \in \mathcal{E}_p$.

First note that the equation $x = f(x)$ can be rewritten as follows

$$Cx^3 - Ax^2 + x - 1 = 0$$

Note that, in general, we may solve the last equation by methods developed in [46, ?]. But those methods give only information about the existence of solutions. In reality, we need more properties of the solutions (see further sections). Therefore, we are going to find all the solutions.

Since x_0 is a solutions of the last equation, therefore, one has

$$(4.9) \quad Cx^3 - Ax^2 + x - 1 = (x - x_0)(Cx^2 + (Cx_0 - A)x + 1/x_0).$$

Let us solve

$$(4.10) \quad Cx^2 + (Cx_0 - A)x + 1/x_0 = 0.$$

From (i) and the conditions of the proposition we can write

$$(4.11) \quad \frac{1}{x_0} = 1 + \epsilon_0 p^{\gamma_0}, \quad C = \epsilon_1 p^{\gamma_1}, \quad A = \epsilon_2 p^{\gamma_2},$$

where $\epsilon_0, \epsilon_2, \epsilon_3 \in \mathbb{Z}_p$ and $\gamma_0, \gamma_1, \gamma_2 > 0$.

- (ii) From $|A|_p^2 < |C|_p$ it follows that $2\gamma_2 > \gamma_1$.

Hence, the discriminant of (4.10) can be calculated as follows

$$\begin{aligned}
 \Delta &= (Cx_0 - A)^2 - \frac{4C}{x_0} \\
 &= p^{\gamma_1} \left(-\frac{4\epsilon_1}{x_0} + \epsilon_1^2 x_0 p^{\gamma_1} - 2\epsilon_1 \epsilon_2 x_0 p^{\gamma_2} + \epsilon_2^2 p^{2\gamma_2 - \gamma_1} \right) \\
 &= p^{\gamma_1} \left(-4\epsilon_1 - 4\epsilon_1 \epsilon_0 p^{\gamma_0} + \epsilon_1^2 x_0 p^{\gamma_1} - 2\epsilon_1 \epsilon_2 x_0 p^{\gamma_2} + \epsilon_2^2 p^{2\gamma_2 - \gamma_1} \right) \\
 (4.12) \quad &= p^{\gamma_1} (-4\epsilon_1 + \tilde{\epsilon} p^{\tilde{\gamma}})
 \end{aligned}$$

for some $\tilde{\epsilon} \in \mathbb{Z}_p$ and $\tilde{\gamma} > 0$.

Consequently, from Lemma 2.1 we conclude that $\sqrt{\Delta}$ exists if and only if $\sqrt{-4C}$ exists, which is equivalent to the existence of $\sqrt{-C}$. So, it follows from (4.15) that $|\sqrt{\Delta}|_p = \sqrt{|C|_p}$.

Assume that (4.10) has two solutions x_1, x_2 , which have the following form

$$(4.13) \quad x_{1,2} = \frac{A - Cx_0 \pm \sqrt{\Delta}}{2C}.$$

Taking into account that

$$(4.14) \quad |A - C|_p \leq \max\{|A|_p, |C|_p\} < \sqrt{|C|_p},$$

and $|C(x_0 + 1)|_p = |C|_p$ with the strong triangle inequality from (4.13) we obtain

$$\begin{aligned}
 |x_{1,2} - 1|_p &= \frac{1}{|C|_p} |A - C - C(x_0 + 1) \pm \sqrt{\Delta}|_p \\
 &= \frac{1}{\sqrt{|C|_p}}.
 \end{aligned}$$

The last equality implies (4.6).

(iii) Now assume that $|A|_p^2 > |C|_p$. This means that $2\gamma_2 < \gamma_1$. Then from (4.15) we find that

$$(4.15) \quad \Delta = A^2(1 + \delta_1 p^{\tilde{\gamma}_1})$$

for some $\delta_1 \in \mathbb{Z}_p$ and $\tilde{\gamma}_1 > 0$. Hence, again from Lemma 2.1 we infer that $\sqrt{\Delta}$ exists. Moreover, one has $\sqrt{\Delta} = A(1 + \delta_2 p^{\tilde{\gamma}_2})$, for some $\delta_2 \in \mathbb{Z}_p$ and $\tilde{\gamma}_2 > 0$. This yields that

$$|A + \sqrt{\Delta}|_p = |A|_p, \quad |A - \sqrt{\Delta}|_p < |A|_p.$$

For the solutions $x_{1,2}$ (see (4.13)) from the last equalities we obtain

$$(4.16) \quad |x_1|_p = \left| \frac{A + \sqrt{\Delta} - Cx_0}{2C} \right|_p = \frac{|A|_p}{|C|_p},$$

since $|A|_p > |A|_p^2 > |C|_p$.

From the equality $x_1 x_2 = \frac{1}{Cx_0}$ with (4.16) one finds

$$|x_2|_p = \frac{1}{|A|_p}.$$

This completes the proof. □

According to Theorem 3.1 the solutions x_0 , x_1 and x_2 (If they exist) generate generalized p -adic quasi Gibbs measures μ_0 , μ_1 and μ_2 , respectively. Hence, we can formulate the following result.

Theorem 4.3. *Let $p \geq 3$, $|\rho|_p < 1$. Assume that for the function λ one has*

$$(4.17) \quad \lambda(1, 1), \lambda(-1, 1) > 0, \quad \lambda(1, -1) = \lambda(-1, -1) = 0.$$

Then for the λ -model (3.1) on the Cayley tree of order two the following assertions hold:

- (i) *there exists a transition-invariant generalized p -adic quasi Gibbs Measure μ_0 ;*
- (ii) *if*

$$2\lambda(1, 1) > \lambda(-1, 1),$$

then there are three transition-invariant generalized p -adic quasi Gibbs measures μ_0 , μ_1 and μ_2 if and only if $\sqrt{-\rho^{\lambda(-1, 1)}}$ exists;

- (ii) *if*

$$2\lambda(1, 1) < \lambda(-1, 1),$$

then there are three transition-invariant generalized p -adic quasi Gibbs measures μ_0 , μ_1 and μ_2 .

In this paper, our main aim to establish the existence of phase transitions for the model. In [41] we have proposed to predict the phase transitions by looking at behavior of the function f . Now we are going to determine behaviors of the fixed points of the function.

Proposition 4.4. *Let $|A|_p, |C|_p < 1$ with $|A|_p \neq |C|_p$, and f be given by (4.4). Then the following statements hold:*

- (i) *The fixed point x_0 is attractive;*
- (ii) *Assume that $|A|_p^2 < |C|_p$ and $\sqrt{-C}$ exists. Then the fixed points $x_{1,2}$ are repelling;*
- (iii) *Assume that $|A|_p^2 > |C|_p$. Then the fixed point x_1 is attractive and x_2 is neutral.*

Proof. From (4.4) we find that

$$(4.18) \quad f'(x) = \frac{2(A - C)x}{(Cx^2 + 1)^2}.$$

(i). Since $x_0 \in \mathcal{E}_p$, from (4.18) we get $|f'(x_0)|_p = |A - C|_p < 1$, which means that x_0 is attractive.

(ii). Assume that $|A|_p^2 < |C|_p$ and $\sqrt{-C}$ exists. Then from Theorem 4.2 we conclude that the fixed points $x_{1,2}$ exist and satisfy the following equality

$$Cx_{1,2}^2 = (A - Cx_0)x_{1,2} - \frac{1}{x_0}.$$

Therefore, we have

$$(4.19) \quad \begin{aligned} |Cx_{1,2}^2 + 1|_p &= \left| (A - Cx_0)x_{1,2} - \frac{1}{x_0} + 1 \right|_p \\ &= \left| (A - C)x_{1,2} - C(x_0 - 1)x_{1,2} + \frac{1 - x_0}{x_0} \right|_p. \end{aligned}$$

From (4.6) and (4.8) it follows that

$$|(A - C)x_{1,2}|_p = \frac{|A - C|_p}{\sqrt{|C|_p}}, \quad |C(x_0 - 1)x_{1,2}|_p = \sqrt{|C|_p}|A - C|_p, \quad |1 - x_0|_p = |A - C|_p.$$

Hence,

$$|(A - C)x_{1,2}|_p > |1 - x_0|_p > |C(x_0 - 1)x_{1,2}|_p.$$

So, the last inequalities together with the strong triangle inequality imply that (4.19) can be calculated as follows

$$(4.20) \quad |Cx_{1,2}^2 + 1|_p = \frac{|A - C|_p}{\sqrt{|C|_p}}.$$

Now from (4.18) with (4.20), (4.6) one gets

$$(4.21) \quad |f'(x_{1,2})|_p = \frac{|A - C|_p|x_{1,2}|_p}{|Cx_{1,2}^2 + 1|_p^2} = \frac{\sqrt{|C|_p}}{|A - C|_p} > 1$$

this implies that $x_{1,2}$ is repelling.

(iii). Assume that $|A|_p^2 > |C|_p$, then the fixed points $x_{1,2}$ exist. Then from (4.7) we immediately find

$$(4.22) \quad |Cx_1^2 + 1|_p = \frac{|A|_p^2}{|C|_p}, \quad |Cx_2^2 + 1|_p = 1.$$

Therefore, from (4.18), (4.22) one gets

$$(4.23) \quad |f'(x_1)|_p = \frac{|C|_p}{|A|_p^2} < 1, \quad |f'(x_2)|_p = 1.$$

This means that x_1 is attractive and x_2 is neutral. The proof is complete. \square

5. PHASE TRANSITIONS

In this section, we are going to establish the existence of the phase transition for λ -models in the considered two regimes.

According to dynamical approach, taking into account Proposition 4.4 we may predict that if $|A|_p^2 < |C|_p$, then there occurs a phase transition, and if $|A|_p^2 > |C|_p$, then there exists a quasi phase transition. In this section, we will confirm that our predictions are true.

Before, going to prove main results we need some auxiliary facts.

Lemma 5.1. [44] *Ler $\rho \in \mathbb{Q}_p$ and \mathbf{h} be a solution of (3.6), and $\mu_{\mathbf{h},\rho}$ be an associated generalized p -adic quasi Gibbs measure. Then for the corresponding partition function $Z_{n,\rho}^{(\mathbf{h})}$ (see (3.3)) the following equality holds*

$$(5.1) \quad |Z_{n+1,\rho}^{(\mathbf{h})}|_p = |A_{\mathbf{h},n}|_p |Z_{n,\rho}^{(\mathbf{h})}|_p,$$

where

$$(5.2) \quad |A_{\mathbf{h},n}|_p = \prod_{x \in W_n} |a(x)|_p,$$

here

$$(5.3) \quad |a(x)|_p^2 = \left| \prod_{y \in S(x)} \sum_{\eta(y) \in \{-1,1\}} \rho^{\lambda(1,\eta(y))} (h_y)^{\eta(y)} \right|_p \left| \prod_{y \in S(x)} \sum_{\eta(y) \in \{-1,1\}} \rho^{\lambda(-1,\eta(y))} (h_y)^{\eta(y)} \right|_p$$

From this lemma we immediately find the following

Lemma 5.2. *Let $\mathbf{h} = \{h_x\}$ be a translation-invariant solution of (3.6), i.e. $h_x = h_*$ for all $x \in V$. Then one has*

$$(5.4) \quad Z_{n,\rho}^{(\mathbf{h})} = \frac{1}{|h_*|_p^{|V_{n-1}|}} \left| \rho^{\lambda(-1,1)} h_*^2 + \rho^{\lambda(-1,-1)} \right|_p^{2|V_{n-1}|}.$$

5.1. Regime $|A|_p^2 < |C|_p$. In this subsection our main result is the following result.

Theorem 5.3. *Let $p \geq 3$, $|\rho|_p < 1$. Assume that for the function λ one has*

$$2\lambda(1,1) > \lambda(-1,1), \quad \lambda(1,-1) = \lambda(-1,-1) = 0.$$

and $\sqrt{-\rho^{\lambda(-1,1)}}$ exists. Then there exist the phase transition for the λ -model (3.1) on the Cayley tree of order two.

Proof. First we note that due to Theorem 4.3 (ii) there are three translation-invariant generalized p -adic Gibbs measures μ_0, μ_1, μ_2 .

Assume that $\mathbf{h} = \{h_x\}$ is a translation-invariant solution of (3.6). Then $h_x = h_*$ for all $x \in V$, where h_* is a fixed point of f . Then due to Lemma 5.2 from (3.2) together with (5.4) we obtain

$$(5.5) \quad |\mu_{n,\rho,*}(\sigma)|_p = \frac{|\rho|_p^{H_n(\sigma)} |h_*|_p^{\sum_{x \in W_n} \sigma(x)} |h_*|_p^{|V_{n-1}|}}{|Ch_*^2 + 1|_p^{2|V_{n-1}|}}.$$

Let us consider the measure μ_0 . Since $x_0 \in \mathcal{E}_p$ (see Proposition 4.2) and $|C|_p < 1$, from (5.5) one gets

$$(5.6) \quad |\mu_{n,0}(\sigma)|_p = |\rho|_p^{H_n(\sigma)} < 1$$

This means that μ_0 is bounded.

Now consider the measure $\mu_{1,2}$. From (5.5) together with (4.20), (4.6) one finds

$$(5.7) \quad |\mu_{n,\rho,1,2}(\sigma)|_p = \frac{|\rho|_p^{H_n(\sigma)} |\sqrt{|C|_p}^{-\sum_{x \in W_n} \sigma(x)}|_p^{|V_{n-1}|}}{|A - C|_p^{2|V_{n-1}|}}$$

Define a configuration $\sigma^{(-)}$ on V_n by

$$\sigma^{(-)}(x) = -1, \quad \forall x \in V_n.$$

Then one can see that $H_n(\sigma^{(-)}) = 0$.

Hence, from (5.7) together with (2.5), (4.14) we have

$$(5.8) \quad \begin{aligned} |\mu_{n,\rho,1,2}(\sigma^{(-)})|_p &= \frac{\sqrt{|C|_p}^{|W_n|} \sqrt{|C|_p}^{|V_{n-1}|}}{|A - C|_p^{2|V_{n-1}|}} \\ &= \sqrt{|C|_p} \left(\frac{\sqrt{|C|_p}}{|A - C|_p} \right)^{2|V_{n-1}|} \rightarrow \infty \quad \text{as } n \rightarrow \infty. \end{aligned}$$

which implies $\mu_{1,2}$ is unbounded.

Hence, (5.6) and (5.8) imply the existence of the phase transition. This completes the proof. \square

Remark 5.1. This proved theorem confirms that if the dynamical system associated with a model has at least two repelling fixed points, then for the model exhibits a phase transition. We stress that the considered λ -model has the stronger phase transition (see [44]).

Remark 5.2. If one takes $\rho = p$ and $\lambda(-1, 1) = 2m$ for some $m \in \mathbb{N}$, then $\sqrt{-p^{2m}}$ exists if and only if $p \equiv 1 \pmod{4}$.

5.2. Regime $|A|_p^2 > |C|_p$. In this subsection we prove the following result.

Theorem 5.4. *Let $p \geq 3$, $|\rho|_p < 1$. Assume that for the function λ one has*

$$(5.9) \quad 2\lambda(1, 1) < \lambda(-1, 1), \quad \lambda(1, -1) = \lambda(-1, -1) = 0.$$

Then there exist the quasi phase transition for the λ -model (3.1) on the Cayley tree of order two.

Proof. Theorem 4.3 (iii) implies the existence of three translation-invariant generalized p -adic Gibbs measures μ_0, μ_1, μ_2 .

We remark that according to Proposition 4.4 the dynamical system f has two attractive and one neutral fixed points. This indicated to the existence of quasi phase transition.

In the considered regime, by the same argument as in the proof of Theorem 5.3 one can establish that the measure μ_0 is also bounded. Moreover, one can find

$$(5.10) \quad |\mu_{n,\rho,1,2}(\sigma)|_p = \frac{|\rho|_p^{H_n(\sigma)} |x_{1,2}|_p^{\sum_{x \in W_n} \sigma(x)} |x_{1,2}|_p^{|V_{n-1}|}}{|Cx_{1,2}^2 + 1|_p^{2|V_{n-1}|}}.$$

Now consider the measure $\mu_{n,\rho,1}$. Then from (5.10), (4.7) and (4.22) we obtain

$$(5.11) \quad \begin{aligned} |\mu_{n,\rho,1}(\sigma)|_p &= \frac{|\rho|_p^{H_n(\sigma)} \left(\frac{|A|_p}{|C|_p} \right)^{\sum_{x \in W_n} \sigma(x) + |V_{n-1}|}}{\left(\frac{|A|_p^2}{|C|_p} \right)^{2|V_{n-1}|}} \\ &= \frac{|\rho|_p^{H_n(\sigma)} |A|_p^{\sum_{x \in W_n} \sigma(x) - 3|V_{n-1}|}}{|C|_p^{\sum_{x \in W_n} \sigma(x) - |V_{n-1}|}}. \end{aligned}$$

Due to $H_n(\sigma) \leq |V_n| - 1$ and $|C|_p < |A|_p$ we find

$$(5.12) \quad |\rho|_p^{H_n(\sigma)} \leq |A|_p^{|V_n|-1}.$$

Taking into account the last expression with

$$-|W_n| \leq \sum_{x \in W_n} \sigma(x) \leq |W_n|,$$

and (2.5) from (5.11) one gets

$$\begin{aligned}
|\mu_{n,\rho,1}(\sigma)|_p &\leq \frac{|A|_p^{|V_n|-3|V_{n-1}|-1+\sum_{x \in W_n} \sigma(x)}}{|C|_p^{\sum_{x \in W_n} \sigma(x)-|V_{n-1}|}} \\
&= \left(\frac{|C|_p}{|A|_p} \right)^{|V_{n-1}|-\sum_{x \in W_n} \sigma(x)} \\
&\leq \left(\frac{|C|_p}{|A|_p} \right)^{|V_{n-1}|-|W_n|} \\
&= \frac{|C|_p}{|A|_p} < 1.
\end{aligned}$$

This means that the measure μ_1 is bounded.

Let us consider the measure $\mu_{n,\rho,2}$. Then from (5.10), (4.7), (4.22), (5.12) one finds

$$\begin{aligned}
(5.13) \quad |\mu_{n,\rho,2}(\sigma)|_p &= |\rho|_p^{H_n(\sigma)} \left(\frac{1}{|A|_p} \right)^{\sum_{x \in W_n} \sigma(x)+|V_{n-1}|} \\
&\leq |A|_p^{|V_n|-1-\sum_{x \in W_n} \sigma(x)-|V_{n-1}|} \\
&= |A|_p^{|V_{n-1}|-\sum_{x \in W_n} \sigma(x)} \\
&\leq |A|_p^{|V_{n-1}|-|W_{n-1}|} \\
&= \frac{1}{|A|_p}
\end{aligned}$$

This means that the measure μ_2 is bounded as well.

Consequently, we infer the existence of the quasi phase transition. This completes the proof. \square

By $\sigma|_{W_n}$ we denote the restriction of a configuration σ to W_n . Define a configurations $\sigma_n^{(\pm)}$ on W_n by

$$\sigma_n^{(\pm)}(x) = \pm 1, \quad \forall x \in W_n.$$

Corollary 5.5. *Let $p \geq 3$, $|\rho|_p < 1$ and assume (5.9) is satisfied. Let*

$$(5.14) \quad A_n^{(\pm)} = \{\sigma \in \Omega_{V_n} : \sigma|_{W_n} = \sigma_n^{(\pm)}\}$$

Then one has

$$(5.15) \quad \left| \frac{\mu_{n,\rho,1}(\sigma)}{\mu_{n,\rho,2}(\sigma)} \right|_p \rightarrow 0, \quad n \rightarrow \infty, \quad \text{for all } \sigma \in A_n^{(-)}$$

$$(5.16) \quad \left| \frac{\mu_{n,\rho,1}(\sigma)}{\mu_{n,\rho,2}(\sigma)} \right|_p \rightarrow \infty, \quad n \rightarrow \infty, \quad \text{for all } \sigma \in A_n^{(+)}.$$

Proof. Take any σ from $A_n^{(-)}$. Then

$$\sum_{x \in W_n} \sigma(x) = -|W_n|,$$

so from (5.11),(5.13) we get

$$\begin{aligned}
 \left| \frac{\mu_{n,\rho,1}(\sigma)}{\mu_{n,\rho,2}(\sigma)} \right|_p &= \frac{|A|_p^{\sum_{x \in W_n} \sigma(x) - 2|V_{n-1}|}}{|C|_p^{\sum_{x \in W_n} \sigma(x) - |V_{n-1}|}} \\
 &= \frac{|A|_p^{-|W_n| - 2|V_{n-1}|}}{|C|_p^{-|W_n| - |V_{n-1}|}} \\
 &= \frac{|C|_p^{2|V_{n-1}|+1}}{|A|_p^{3|V_{n-1}|+1}} \\
 (5.17) \qquad &= \frac{|C|_p}{|A|_p} \left(\frac{|C|_p^2}{|A|_p^3} \right)^{|V_{n-1}|}.
 \end{aligned}$$

From $|C|_p < |A|_p^2$ we infer that $|C|_p^2 < |A|_p^4 < |A|_p^3$, which with (5.17) implies (5.15).

Now take $\sigma \in A_n^{(+)}$. Then using the same argument as above one finds

$$\begin{aligned}
 \left| \frac{\mu_{n,\rho,1}(\sigma)}{\mu_{n,\rho,2}(\sigma)} \right|_p &= \frac{|A|_p^{|W_n| - 2|V_{n-1}|}}{|C|_p^{|W_n| - |V_{n-1}|}} \\
 &= \frac{|A|_p^{1 - |V_{n-1}|}}{|C|_p} \\
 (5.18) \qquad &\geq \frac{|A|_p}{|A|_p^{|V_{n-1}|}} \rightarrow \infty \text{ as } n \rightarrow \infty.
 \end{aligned}$$

This completes the proof. □

Remark 5.3. This corollary shows that the bounded measures μ_1 and μ_2 are "singular" on the sets $A_n(\pm)$, which yields that they are different from each other.

6. CONCLUSIONS

It is known that in the investigate of phase transitions the renomalization method is one of the powerful tools in theoretical and mathematical physics. In real case, this method has greatly enhanced our understanding of complex properties of models. The interplay of statistical mechanics with chaos theory has even led to novel conceptual frameworks in different physical settings [13]. Therefore, in the present paper, we have proposed to investigate phase transition phenomena from renomalization technique perspective. In the paper, we considered p -adic λ -model on the Cayley tree. Note that if one takes $\lambda(x, y) = Nxy$, then such a model reduces to the Ising model. This model was studied in [44, 41]. But in the paper, we have concentrated ourselves to a totally different model than the Ising one. For such a model, we have considered two regimes with respect to a parameters A and C . It was proved the existence of generalized p -adic Gibbs measures in both regimes. We obtained a p -adic dynamical system and investigate its fixed points. In the first regime we establish that the dynamical system has one attractive and two repelling fixed points, which predicts the existence of a phase transition. In the second regime the system has two attractive and one neutral fixed points, which predicts the existence of a quasi phase transition. Main results of the present paper are to verify (i.e. rigorously prove) and confirm that the indicated predictions (via dynamical systems point of view) are

indeed true. These investigations show that there are some similarities with the real case, for example, the existence of two repelling fixed points implies the occurrence of the phase transition. But there are also some differences. Namely, when the dynamical system has two attractive fixed points, there occurs quasi phase transition, unlike in real case, there is not such kind of behavior. Finally, using such a method one can study other p -adic models over trees.

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